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Stability analysis and quasi-neutral limit for the Euler-Poisson equations

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1 Introduction

The purpose of this survey paper is to mathematically discuss the formation of a plasma sheath near the surface of materials immersed in a plasma, and to study qualitative information of such a plasma sheath layer. In fact we summarize the results [3–5, 9–11] investigating the asymptotic behavior and quasi-neutral limit of solutions to the Euler-Poisson equations in a half space or three-dimensional annular domain.

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The motion of positive ions in a plasma is governed by, after suitable nondimensionalization, the Euler-Poisson equations:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1a}$$

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) + K\nabla\rho = \rho\nabla\phi, \tag{1b}$$

$$\varepsilon\Delta\phi = \rho - e^{-\phi}, \tag{1c}$$

where the unknown functions $\rho(x,t)$, $u(x,t)$ and $-\phi(x,t)$ represent the density, velocity of the positive ion and the electrostatic potential, respectively. Moreover, $K > 0$ is a constant of temperature of the positive ion, and $\sqrt{\varepsilon}$ is the ratio of the *Debye length* λ_D to the reference spatial scale. The first equation describes the mass balance law, and the second one is the equation of momentum in which the pressure gradient and electrostatic potential gradient as well as the convection effect are taken into account. The third equation is the Poisson equation, which describes the relation between the potential and the density of charged particles. For this equation, the Boltzmann relation stating that the electron density is given by $\rho_e = e^{-\phi}$ is assumed.

The plasma sheath appears when a material is immersed in a plasma and the plasma contacts with its surface. Since the thermal velocities of electrons are much higher than those of ions, more electrons tend to hit the surface of the material than ions do, which makes the material negatively charged with respect to the surrounding plasma. Then the material with a negative potential attracts and accelerates ions toward the surface, while repelling electrons away from it, and this results in the formation of non-neutral potential region near the surface, where a nontrivial equilibrium of the densities is achieved. Consequently, positive ions outnumber electrons in this region and this ion-rich layer near the boundary is referred to as the *plasma sheath.* This boundary layer shields the plasma from the negatively charged material. The thickness of this layer is the same order of the Debye length λ_D. For the formation of sheath, Langmuir in [7] observed that positive ions must enter the sheath region with a sufficiently large kinetic energy. Bohm in [1] derived the original *Bohm criterion* for the plasma containing electrons and only one component of mono-valence ions, which states that the ion velocity u at the plasma edge must exceed the ion acoustic speed for the case of planar wall. For more details of the sheath formation, we refer the readers to [2, 8, 12].

This paper is organized as follows. In Section 2, we give an overview of mathematical research [3, 4, 9–11] studying the Euler-Poisson equations (1) in the half space. The papers [9–11] show the asymptotic stability and unique existence of stationary solutions to (1) by assuming the Bohm criterion. These mathematically justify the Bohm criterion and ensure that the sheath corresponds to the stationary solution. Furthermore, we briefly discuss the quasi-neutral limit ($\varepsilon \to 0$) of time-local solutions to (1) studied in [3, 4]. Section 3 provides our recent results [5] analyzing the Euler-Poisson equations (1) in the three-dimensional annular domain, for which we propose the *Bohm criterion for the* *annulus* and prove the unique existence of stationary spherical symmetric solutions under this Bohm criterion. We also study the quasi-neutral limit behavior of this stationary solution by establishing L^2 and H^1 estimates of the difference between the solutions to (1) and its quasi-neutral limiting equations, incorporated with the correctors for the boundary layers. The pointwise estimate of correctors enables us to obtain the thickness of boundary layers. In the last section, we give concluding remarks for the above results and related issues. Before closing the introduction, we present several notations to be used throughout the paper.

Notation. For a non-negative integer l, $H^l(\Omega)$ denotes the l-th order Sobolev space in the L^2 sense. We note $H^0 = L^2$ and $H^\infty = \bigcap_{l\geq 0} H^l$. For a non-negative integer k, $C^k([0,T]; H^l(\Omega))$ denotes the space of k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\Omega)$. We note $C^\infty([0,T]; H^\infty(\Omega)) = \bigcap_{l,k\geq 0} C^k([0,T]; H^l(\Omega))$. Furthermore, C and c are generic positive constants.

2 Half space problem

This section is devoted to discuss the previous researches of the Euler-Poisson equations (1) in the three-dimensional half space

$$\mathbb{R}^3_+ := \{x = (x_1, x') = (x_1, x_2, x_3) \in \mathbb{R}^3; x_1 > 0\}$$

with the initial and boundary data

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \quad \inf_{x\in\mathbb{R}^3_+} \rho(x) > 0, \quad \lim_{x_1\to\infty} (\rho_0, u_0)(x_1, x') = (1, u_+, 0, 0), \tag{2}$$

$$\phi(t, 0, x') = \phi_b, \quad \lim_{x_1\to\infty} \phi(t, x_1, x') = 0. \tag{3}$$

The papers [9,10] proved the uniquè existence and asymptotic stability of planer stationary solutions $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ to problem (1)–(3) by assuming the original Bohm criterion

$$u_+^2 > K + 1, \quad u_+ < 0. \tag{4}$$

Here the planar stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1) = (\tilde{\rho}, \tilde{u}_1, 0, 0, \tilde{\phi})(x_1)$ is a solution to (1) independent of the time variable t and tangential variable x'. Therefore, it satisfies

$$(\tilde{\rho}\tilde{u}_1)_{x_1} = 0, \tag{5a}$$

$$\left(\tilde{\rho}\tilde{u}_1{}^2 + K\tilde{\rho}\right)_{x_1} = \tilde{\rho}\tilde{\phi}_{x_1}, \tag{5b}$$

$$\varepsilon\tilde{\phi}_{x_1x_1} = \tilde{\rho} - e^{-\tilde{\phi}} \tag{5c}$$

with the same conditions as in (2) and (3), that is,

$$\inf_{x_1\in\mathbb{R}_+} \tilde{\rho}(x_1) > 0, \quad \lim_{x_1\to\infty} (\tilde{\rho}, \tilde{u}_1, \tilde{\phi})(x_1) = (\rho_+, u_+, 0), \quad \tilde{\phi}(0) = \phi_b. \tag{5d}$$

In the analysis of stationary solutions, we employ the Sagdeev potential V defined by

$$V(\phi) := \int_0^{\phi} f^{-1}(\eta) - e^{-\eta}\, d\eta, \quad f(\rho) := K \log \rho + \frac{u_+^2}{2\rho^2} - \frac{u_+^2}{2},$$

where the domain of f is an interval $I := (0, |u_+|/\sqrt{K}]$.

The results in [9, 10] are summarized in the following three theorems.

Theorem 2.1 ([10]). *Let* (4) *hold. The boundary data ϕ_b satisfies $V(\phi_b) \geq 0$ and $\phi_b \leq f(|u_+|/\sqrt{K})$ if and only if stationary problem* (5) *has a unique monotone solution* $(\tilde{\rho}, \tilde{u}_1, \tilde{\phi}) \in C(\overline{\mathbb{R}_+}) \cap C^2(\mathbb{R}_+)$.

Theorem 2.2 ([9, 10]). *Let* (4) *hold. Suppose* $(e^{\alpha x_1/2}(\rho_0 - \tilde{\rho}), e^{\alpha x_1/2}(u_0 - \tilde{u})) \in H^3(\mathbb{R}^3_+)$ *for some positive constant α. Then there exist positive constants $\beta(\leq \alpha)$ and δ such that if $|\phi_b| + \|(e^{\beta x_1/2}(\rho_0 - \tilde{\rho}), e^{\beta x_1/2}(u_0 - \tilde{u}))\|_{H^3} \leq \delta$, initial-boundary value problem* (1)–(3) *has a unique time-global solution*

$$\left(e^{\beta x_1/2}(\rho - \tilde{\rho}), e^{\beta x_1/2}(u - \tilde{u}), e^{\beta x_1/2}(\phi - \tilde{\phi})\right) \in \bigcap_{j=0}^{3} C^j([0,\infty); H^{3-j}(\mathbb{R}^3_+)).$$

Moreover, it satisfies the decay estimate

$$\sup_{x \in \mathbb{R}^3_+} |(\rho - \tilde{\rho}, u - \tilde{u}, \phi - \tilde{\phi})(t)| \leq Ce^{-\gamma t},$$

where positive constants C and γ are independent of the time variable t.

Theorem 2.3 ([9, 10]). *Let* (4) *hold. Suppose* $((1+\alpha x_1)^{\lambda/2}(\rho_0 - \tilde{\rho}), (1+\alpha x_1)^{\lambda/2}(u_0 - \tilde{u})) \in H^3(\mathbb{R}^3_+)$ *for some $\lambda \geq 2$ and $\alpha > 0$. Then there exist positive constants $\beta(\leq \alpha)$ and δ such that if $|\phi_b| + \|((1+\beta x_1)^{\lambda/2}(\rho_0 - \tilde{\rho}), (1+\beta x_1)^{\lambda/2}(u_0 - \tilde{u}))\|_{H^3} \leq \delta$, initial-boundary value problem* (1)–(3) *has a unique time-global solution*

$$\left((1+\beta x_1)^{\lambda/2}(\rho - \tilde{\rho}), (1+\beta x_1)^{\lambda/2}(u - \tilde{u}), (1+\beta x_1)^{\lambda/2}(\phi - \tilde{\phi})\right) \in \bigcap_{j=0}^{3} C^j([0,\infty); H^{3-j}(\mathbb{R}^3_+)).$$

Moreover, it satisfies the decay estimate

$$\sup_{x \in \mathbb{R}^3_+} |(\rho - \tilde{\rho}, u - \tilde{u}, \phi - \tilde{\phi})(t)| \leq C(1+\beta t)^{-(\lambda - \zeta)},$$

for any $\zeta \in (0, \lambda]$, where positive constants C is independent of the time variable t.

Some similar results are obtained in [11] for a multicomponent plasma containing electrons and several components of ions under the generalized Bohm criterion derived in [13]. These results give a mathematical validity for the Bohm criterion and ensure that the sheath corresponds to the stationary solution.

The quasi-neutral limit ($\varepsilon \to 0$) of time-local solutions to (1)–(3) is studied by D. Gérard-Varet, D. Han-Kwan and F. Rousset. This limit problem is considered under the original Bohm criterion at the boundary $\{x_1 = 0\}$ in [4], and some other cases without the Bohm criterion are also studied in [3]. In particular, the former paper gives the H^k-estimates of the difference of the solutions to the Euler-Poisson equations and its quasi-neutral limiting equations, incorporated with the correctors for the boundary layers.

The limiting equations are defined by setting $\varepsilon = 0$ in (1) as

$$\begin{gathered} \rho^0_t + \nabla \cdot (\rho^0 u^0) = 0, \\ (\rho^0 u^0)_t + \nabla \cdot \left(\rho^0 u^0 \otimes u^0\right) + K\nabla \rho^0 = \rho^0 \nabla \phi^0, \\ \rho^0 - e^{-\phi^0} = 0. \end{gathered}$$

We prescribe the initial data $(\rho^0_0, u^0_0) \in H^\infty(\mathbb{R}^3_+)$ satisfying

$$\begin{gathered} \inf_{x\in\mathbb{R}^3_+} \rho^0_0(x) > 0, \quad \sup_{x\in\mathbb{R}^3_+} u^0_{30}(x) < 0, \quad (\sup_{x\in\mathbb{R}^3_+} u^0_{30}(x))^2 > K+1, \\ \lim_{x_1\to\infty} (\rho_0, u_0)(x_1, x') = (1, u_+, 0, 0) \end{gathered}$$

and no boundary condition. It is seen from Appendix of [6] that this initial–boundary value problem of limiting equations admits a unique time-local solution

$$\left(\rho^0 - 1, u^0_1 - u_+, u^0_2, u^0_3\right) \in C^\infty([0,T]; H^\infty(\mathbb{R}^3_+))$$

for some $T > 0$.

Note that it is not acceptable to impose the boundary condition for ϕ^0 for the limiting equations since it is determined by the one for ρ^0, whereas the boundary condition for ϕ is necessary for the problem (1)–(3). In general, the boundary data ϕ_b does not satisfy the relation $\phi^0(t, 1, x') = \phi_b$. Hence, one can expect in the analysis of the quasi-neutral limit that this discrepancy of the boundary data would cause a steep change in the solution near the boundary $\{x_1 = 0\}$. This sharp transition near the surface is referred to as a boundary layer. To handle this, we need the correctors for the boundary layers $(\theta_\rho, \theta_u, \theta_\phi)(t, x_1, x') = (R^0, U^0, 0, 0, \Phi^0)(t, x_1/\sqrt{\varepsilon}, x') \in C^\infty([0,T]; H^\infty(\mathbb{R}^3_+))$, which are the solutions of ordinary differential equations

$$\begin{gathered} \left\{(\Gamma\rho^0 + R^0)(\Gamma u^0_1 + U^0)\right\}_{x_1} = 0, \\ \frac{1}{2}\{(\Gamma u^0_1 + U^0)^2\}_{x_1} + \frac{K R^0_{x_1}}{\Gamma\rho^0 + R^0} = \Phi^0_{x_1}, \\ \Phi^0_{x_1 x_1} = \Gamma\rho^0 + R^0 - e^{\Gamma\phi^0 + \Phi^0} \end{gathered}$$

with the boundary conditions

$$\Phi^0(0) = \phi_b, \quad \lim_{x_1\to\infty} (R^0, U^0, \Phi^0)(x_1) = (0, 0, 0),$$

where Γ is the trace operator at $\{x_1 = 0\}$. The solvability of this boundary value problem can be proved similarly as Theorem 2.1.

The result of D. Gérard-Varet, D. Han-Kwan and F. Rousset is stated in Theorem 2.4. They make use of the approximate expansion in powers of $\sqrt{\varepsilon}$ up to order m:

$$(\rho^m_{app}, u^m_{app}, \phi^m_{app})(t,x) = \sum_{i=0}^{m} \sqrt{\varepsilon}^i (\rho^i, u^i, \phi^i)(t,x) + \sum_{i=0}^{m} \sqrt{\varepsilon}^i (R^i, U^i, \Phi^i)\left(t, \frac{x_1}{\sqrt{\varepsilon}}, x'\right)$$

for any non-negative integer m, where (ρ^i, u^i, ϕ^i) and (R^i, U^i, Φ^i) are the outer and inner solutions, respectively, (the definitions of the outer and inner solutions are omitted in [4] since they are standard), and they also justify this approximate expansion for higher orders $m \geq 1$ in Theorem 3 of [4].

Theorem 2.4 ([4]). *Let the initial data (ρ_0, u_0) satisfy $(\rho_0, u_0)(x) = (\rho^3_{app}, u^3_{app})(0,x) + \varepsilon^2(h_1, h_2)(x)$ for some $(h_1, h_2) \in H^3(\mathbb{R}^3_+)$ independent of ε. There exits some positive constants δ_0 and ε_0 such that if $\varepsilon \in (0, \varepsilon_0]$ and $\sup_{x' \in \mathbb{R}^2} |\rho^0_0(0,x') - e^{-\phi_b}| \leq \delta_0$, initial-boundary value problem* (1)–(3) *admits a unique time-local solution*

$$(\rho^\varepsilon - 1, u^\varepsilon_1 - u_+, u^\varepsilon_2, u^\varepsilon_3) \in \bigcap_{j=0}^{3} C^j([0,T]; H^{3-j}(\mathbb{R}^3_+)),$$

where $T > 0$ is a positive constant independent of ε. Moreover, it satisfies

$$\sup_{0 \leq t \leq T} \|(\rho^\varepsilon - \rho^0 - \theta_\rho, u^\varepsilon - u^0 - \theta_u, \phi^\varepsilon - \phi^0 - \theta_\phi)(t)\|_{L^\infty} \to 0.$$

We emphasize that the thickness of sharp transition of the corrector for the boundary layer $(\theta_\rho, \theta_u, \theta_\phi)$ is of order of $\sqrt{\varepsilon}$ since (R^0, U^0, Φ^0) decays exponentially fast as $x_1 \to \infty$. This thickness corresponds to the physical observation of sheaths. Furthermore, it is seen by formal computations that the corrector $(\theta_\rho, \theta_u, \theta_\phi)$ converges to the planer stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ in Theorem 2.1 as $t \to \infty$. Therefore, the boundary layer, referred in the mathematical sense, of the stationary problem is just given by the planer stationary solution. This validates that the sheaths are the stationary solutions to (1).

3 Annular domain problem

In this section, we consider the stationary problem associated with (1) in the three-dimensional annular domain

$$\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 |\, 1 < |x| < 1 + L\}, \quad L > 0.$$

Here one can set the radii of the inner and outer balls as $R_i = 1$ and $R_o = 1 + L$ by choosing the original radius of the inner ball as the reference spatial scale. The boundary conditions are prescribed as

$$(\rho, u, \phi)(t, x) = (1, u_+ x/|x|, 0) \quad \text{for } x \in \{x \in \mathbb{R}^3 : |x| = 1 + L\}, \tag{6}$$

$$\phi(t, x) = \phi_b \quad \text{for } x \in \{x \in \mathbb{R}^3 : |x| = 1\}, \tag{7}$$

where $u_+ < 0$ and $\phi_b \in \mathbb{R}$ are given constants. For given these spherical symmetric boundary conditions, we seek the corresponding spherical symmetric solutions using the ansatz: Let $\tilde{\rho}(r) = \rho(x)$, $\tilde{u}(r) = u(x) \cdot x/|x|$ and $\tilde{\phi}(r) = \phi(x)$, where $r = |x|$, be the solutions to the boundary value problem of (1), (6) and (7). Here and hereafter f' denotes df/dr. Then the system leads to

$$(\tilde{\rho}\tilde{u})' = -\frac{2}{r}\tilde{\rho}\tilde{u}, \tag{8a}$$

$$\left(\tilde{\rho}\tilde{u}^2 + K\tilde{\rho}\right)' = \tilde{\rho}\tilde{\phi}' - \frac{2}{r}\tilde{\rho}\tilde{u}^2, \tag{8b}$$

$$\varepsilon\frac{1}{r^2}\left(r^2\tilde{\phi}'\right)' = \tilde{\rho} - e^{-\tilde{\phi}} \tag{8c}$$

for $r \in I := (1, 1 + L)$ with the conditions

$$\inf_{r \in I} \tilde{\rho}(r) > 0, \tag{8d}$$

$$(\tilde{\rho}, \tilde{u})(1 + L) = (1, u_+), \tag{8e}$$

$$\tilde{\phi}(1) = \phi_b, \quad \tilde{\phi}(1 + L) = 0. \tag{8f}$$

To discuss the unique existence and quasi-neutral limit behavior of solutions to the stationary problem (8), we first consider the quasi-neutral limiting equations obtained from (8) by setting $\varepsilon = 0$:

$$(\tilde{\rho}^0\tilde{u}^0)' = -\frac{2}{r}\tilde{\rho}^0\tilde{u}^0, \tag{9a}$$

$$\left(\tilde{\rho}^0(\tilde{u}^0)^2 + K\tilde{\rho}^0\right)' = \tilde{\rho}^0\tilde{\phi}^{0\prime} - \frac{2}{r}\tilde{\rho}^0(\tilde{u}^0)^2, \tag{9b}$$

$$\tilde{\rho}^0 - e^{-\tilde{\phi}^0} = 0. \tag{9c}$$

It is shown that the limiting problem of (9), (8d) and (8e) has a unique solution if and only if either

$$u_+^2 \leq (K + 1)a_* \quad \text{or} \quad u_+^2 \geq (K + 1)a^* \tag{10}$$

holds, where a_* and a^* are two roots of the equation $1 + 4\log(1 + L) - x + \log x = 0$ with $a_* < 1 < a^*$. Throughout this section, we assume that

$$u_+^2 > (K + 1)a^*, \quad u_+ < 0 \tag{11}$$

hold. Note that this condition is stronger than the Bohm criterion $u_+^2 > K + 1$ for the planar wall case. We refer to (11) as *the Bohm criterion for the annulus.* It is reasonable to propose this criterion since, under the former condition in (10), the time dependent problem to (1) with (6) and (7) is ill-posed and some research groups of plamsa physics define the Bohm criterion by excluding the equality. The solvability of the limiting problem is summarized in the next lemma.

Lemma 3.1 ([5]). *Let* (11) *hold. Then boundary value problem of* (9), (8d) *and* (8e) *has a unique solution* $(\tilde{\rho}^0, \tilde{u}^0, \tilde{\phi}^0) \in C(\overline{I}) \cap C^\infty(I)$. *Moreover,* $\tilde{\rho}^0$, $\tilde{u}^0$ *and* $-\tilde{\phi}^0$ *are monotonically decreasing functions.*

The stationary problem (8) is also solvable under (11). For the analysis, we employ a function

$$G(r, \tilde{\rho}) := \frac{1}{2}\frac{(1+L)^4 u_+^2}{r^4 \tilde{\rho}^2} + K \log \tilde{\rho} - \frac{1}{2}u_+^2.$$

Here the domain of G is restricted into $D(G) := \overline{I} \times (0, \tilde{\rho}_*(r)]$, where

$$\tilde{\rho}_*(r) := \sqrt{(1+L)^4 u_+^2 r^{-4} K^{-1}}$$

is a critical point of G for each $r \in \overline{I}$. We also make use of the inverse function of G as

$$\tilde{\rho} = g(r, \tilde{\phi}) \quad \text{for} \quad (r, \tilde{\phi}) \in D(g) := \overline{I} \times [G(r, \tilde{\rho}_*(r)), \infty)$$

by solving the equation $G(r, \tilde{\rho}) = \tilde{\phi}$ for $\tilde{\rho}$. We present the unique existence of the stationary solution in the next theorem.

Theorem 3.2 ([5]). *Let* (11) *hold. Then there exist positive constants* ε_1 *and* δ_1 *such that if* $\varepsilon \le \varepsilon_1$ *and* $\phi_b \in [G(1, \tilde{\rho}_*^0(1)), \tilde{\phi}^0(1) + \delta_1)$, *then boundary value problem* (5) *has a unique solution* $(\tilde{\rho}, \tilde{u}, \tilde{\phi}) \in C(\overline{I}) \cap C^\infty(I)$ *satisfying*

$$\tilde{\phi}(r) \ge G(r, \tilde{\rho}_*^0(r)) \quad \textit{for} \quad r \in \overline{I},$$

where $\tilde{\rho}_*^0(r) := \sqrt{(1+L)^4 u_+^2 (K+1)^{-1} r^{-4}}$ *is a critical point of* $F(r, \tilde{\rho}) := G(r, \tilde{\rho}) + \log \tilde{\rho}$.

We are now in a position to analyze the quasi-neutral limit of the stationary spherical symmetric solution constructed in Theorem 3.2. As we mentioned in Section 2, the boundary layers are present in this case. Let us define a corrector of the boundary layer for $\tilde{\phi}$, denoted by θ_ϕ^0, by solving the boundary value problem

$$\varepsilon(\theta_\phi^0)'' = g(r, \tilde{\phi}^0 + \theta_\phi^0) - e^{-\tilde{\phi}^0 - \theta_\phi^0}, \quad \theta_\phi^0(1) = \phi_b - \tilde{\phi}^0(1), \quad \theta_\phi^0(1+L) = 0,$$

where $\tilde{\phi}^0$ is the solution to limiting problem (9). Note that this boundary value problem is solvable under the same assumptions as in Theorem 3.2. The corresponding correctors for $\tilde{\rho}$ and $\tilde{u}$ are given by

$$\theta_\rho^0 := g(r, \tilde{\phi}^0 + \theta_\phi^0) - g(r, \tilde{\phi}^0), \quad \theta_u^0 := -\frac{\tilde{u}^0 \theta_\rho^0}{\tilde{\rho}^0 + \theta_\rho^0},$$

where $\tilde{\rho}^0$ and $\tilde{u}^0$ are the density and velocity for limiting problem (9). We now present the pointwise estimates for $\theta^0_\phi, \theta^0_\rho$ and θ^0_u and the convergence result.

Proposition 3.3 ([5]). *Under the same assumptions as in Theorem 2.1, for any $r \in \bar{I}$, there hold*

$$|\phi_b - \tilde{\phi}^0(1)|\left(e^{-\sqrt{C_0/\varepsilon}(r-1)} - e^{-\sqrt{C_0/\varepsilon L}}\right) \le |\theta^0_\phi(r)| \le |\phi_b - \tilde{\phi}^0(1)|e^{-\sqrt{c_0/\varepsilon}(r-1)},$$
$$c|\phi_b - \tilde{\phi}^0(1)|\left(e^{-\sqrt{C_0/\varepsilon}(r-1)} - e^{-\sqrt{C_0/\varepsilon L}}\right) \le |(\theta^0_\rho, \theta^0_u)(r)| \le C|\phi_b - \tilde{\phi}^0(1)|e^{-\sqrt{c_0/\varepsilon}(r-1)},$$

where c, C, c_0 and C_0 are positive constants independent of ε.

Theorem 3.4 ([5]). *Under the same assumptions as in Theorem 2.1, the difference $(\tilde{\rho} - \tilde{\rho}^0 - \theta^0_\phi,\ \tilde{u} - \tilde{u}^0 - \theta^0_u,\ \tilde{\phi} - \tilde{\phi}^0 - \theta^0_\phi)$ satisfies the decay estimates*

$$\varepsilon^{i/2}\|(\tilde{\rho} - \tilde{\rho}^0 - \theta^0_\rho, \tilde{u} - \tilde{u}^0 - \theta^0_u, \tilde{\phi} - \tilde{\phi}^0 - \theta^0_\phi)\|_{H^i} \le C\varepsilon^{3/4} \quad \textit{for } i = 0, 1,$$
$$\|(\tilde{\rho} - \tilde{\rho}^0 - \theta^0_\rho, \tilde{u} - \tilde{u}^0 - \theta^0_u, \tilde{\phi} - \tilde{\phi}^0 - \theta^0_\phi)\|_{L^\infty} \le C\varepsilon^{1/2},$$

where $C > 0$ is a constant independent of ε.

Proposition 3.3 implies that the correctors in the region away from the inner boundary decay exponentially fast (of order $e^{-\sqrt{1/\varepsilon}}$) in the pointwise sense as $\varepsilon \to 0$, by which one can deduce that the thickness of the boundary layer is of order $\sqrt{\varepsilon}$. This justifies the heuristic explanation in the context of physics stating that the sheath is of several Debye lengths thick.

4 Concluding remarks

Let us mention some concluding remarks by comparing the Theorems in Sections 2 and 3. We first infer from Theorems 2.1 and 3.1 that the Bohm criterion may vary with the geometry of domain. It is expected that the Bohm criterion for the annuals (11) converges to the original Bohm criterion (4) as $\varepsilon \to 0$ since the inner ball of the annulus can locally be regarded as a planer wall when $\varepsilon \ll 1$. However, this expectation turns out to be incorrect since (11) is independent of ε. Therefore, these two criteria are essentially different.

Theorems 2.4 and 3.4 demonstrate that the thickness of the boundary layer is order of $\sqrt{\varepsilon}$ for both the planer wall case and the ball-shaped wall case. The thickness is independent of the geometry of domain. On the other hand, the correctors for the boundary layers which are employed in Theorems 2.4 and 3.4, are defined differently. Therefore, it is not straightforward to compare these two correctors, and it is not clear how their properties are different. This is an interesting question to be investigated for our future work.

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